

PERTURBING MISIUREWICZ PARAMETERS IN THE EXPONENTIAL FAMILY

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ABSTRACT. In one-dimensional real and complex dynamics, a map whose post-singular (or post-critical) set is bounded and uniformly repelling is often called a Misiurewicz map. In results hitherto, perturbing a Misiurewicz map is likely to give a non-hyperbolic map, as per Jakobson's Theorem for unimodal interval maps. This is despite genericity of hyperbolic parameters (at least in the interval setting). We show the contrary holds in the complex exponential family $z \mapsto \lambda \exp(z)$: Misiurewicz maps are Lebesgue density points for hyperbolic parameters.

1. INTRODUCTION

Jakobson's Theorem ([14]) from 1981 is one of the more celebrated and striking results in dynamical systems. In the real quadratic (or logistic) family $f_a : x \mapsto ax(1-x)$, Jakobson showed that there is a positive measure set of parameters a close to the Chebyshev parameter $a = 4$ for which the map has an absolutely continuous, f_a -invariant probability measure μ_a . One can contrast this with the result ([12, 17]), due to Graczyk and Świątek, and to Lyubich, which states that the set of hyperbolic parameters is open and dense, to emphasise the intricacy of quadratic dynamics. Rees in [21] generalised Jakobson's result to rational maps of the Riemann sphere. Benedicks and Carleson extended these results to the Hénon family in [4]. In these settings, one starts with a map with a repelling post-critical set, and sufficiently small perturbations are likely to give non-hyperbolic parameters. In this paper we present a counter-example to this paradigm in the complex exponential family.

In the exponential family $f_\lambda : z \mapsto \lambda e^z$, a parameter λ is called a *Misiurewicz* parameter if $\{f_\lambda^n(0) : n \geq 0\} \subset \mathbb{C}$ is a bounded, hyperbolic repelling set. For Misiurewicz parameters, the Julia set is the entire complex plane (or, regarding f as a meromorphic map, the Julia set is the entire Riemann sphere). In particular, there are dense orbits.

A parameter λ is called *hyperbolic* if f_λ has an attracting periodic orbit. For hyperbolic λ , almost every orbit is in the basin of attraction of the attracting periodic orbit.

Main Theorem. *In the complex exponential family, Misiurewicz parameters are Lebesgue density points for the set of hyperbolic parameters.*

By this we mean, if λ is a Misiurewicz parameter, H is the set of hyperbolic parameters and m denotes Lebesgue measure, then

$$\lim_{r \rightarrow 0^+} \frac{m(B(\lambda, r) \cap H)}{m(B(\lambda, r))} = 1.$$

Misiurewicz parameters (and maps) have a long and involved history in the field of one-dimensional dynamics. Introduced by Misiurewicz in [19] for smooth maps of the interval, they became the first examples where some non-trivial condition on the behaviour of critical orbits guaranteed the existence of absolutely continuous invariant probability measures. This result was superseded by many more in interval dynamics, see [6] for one of the latest and strongest. The concept of Misiurewicz parameter exists in other contexts too, see [13, 5, 2, 9] for example. The articles [14, 24, 21, 3] all find positive measure sets of non-hyperbolic parameters (admitting absolutely continuous invariant probability measures) in a neighbourhood of Misiurewicz parameters. On the other hand, Misiurewicz parameters have zero Lebesgue measure, in general ([23, 1, 2]).

In [24], Thunberg finds positive measure sets of non-hyperbolic parameters in unimodal families of interval maps with critical points of type $\exp(-|x|^{-\alpha})$, provided $\alpha < 1/8$. We showed in [7] that if $\alpha \geq 1$, no absolutely continuous invariant probability measure with positive entropy can exist, as was shown for Misiurewicz parameters in the same setting in [5]. For Misiurewicz parameters in the exponential family, it was proven independently in [9, 15] that no absolutely continuous invariant probability measure can exist.

Structural instability of Misiurewicz parameters in the exponential family was shown in [18, 25, 11]. For the (non-Misiurewicz) map $z \mapsto e^z$, the orbit of 0 is a (wild) metric attractor attracting almost every orbit ([20, 16]), although generic orbits are dense. This map is a density point for hyperbolic maps in the exponential family [26]. For those interested in the structure of parameter space of the exponential family (as opposed to metric properties), we refer to [22].

For a map f_λ from the exponential family, $f_\lambda(z) = Df_\lambda(z)$ and $|f_\lambda(z)| = |\lambda|e^{\Re(z)}$, so f_λ is $2\pi i$ -periodic, f_λ maps vertical lines to circles, horizontal lines to rays emanating from 0, and rectangles of height 2π onto annuli centred at 0. Points far to the left get mapped extremely close to 0, and points far to the right get mapped extremely far from 0.

The proof of the Main Theorem comprises of a number of steps. We show that for an exponential Misiurewicz map the derivative grows exponentially fast, except when it is slowed by the occasional passage close to zero. This allows us to estimate how long it takes for a large proportion of points to make a first entry into a left half-plane (and thus, on the subsequent step, a first entry to a neighbourhood of zero), together with derivative estimates for the first entry map. Next we show that this large proportion of points does not move too fast as the parameter moves. Using the estimates of Badeńska ([2]), for small annuli A_r of parameters λ around the base parameter, $\{f_\lambda^{n_r}(0)\}_{\lambda \in A_r}$ sweeps out a large annulus in phase space, and, for many parameters λ , $f_\lambda^n(0)$ lands on a point for which we have good estimates for the first entry to our neighbourhood of 0. If τ is the first entry time to the neighbourhood, $|Df_\lambda^\tau(0)| \ll 1$ and, controlling distortion, a ball centred on 0 gets mapped inside itself, so an attracting cycle exists.

2. DERIVATIVE ESTIMATES

Throughout the paper, let $f = f_{\lambda_0} : z \mapsto \lambda_0 \exp(z)$, for some $\lambda_0 \in \mathbb{C}$, and suppose that the post-singular set $P(f) := \overline{\{f^n(0) : n \geq 0\}}$ is a bounded hyperbolic repelling set, so there are $n_0, \alpha > 0$ such that $|Df^{n_0}(z)| > \exp(\alpha)$ for all $z \in P(f)$.

Lemma 1. *The Julia set is $\overline{\mathbb{C}}$.*

Proof. This follows immediately from Theorems 3-5 of [10], since the post-singular set is uniformly repelling. \square

Lemma 2. $|\lambda_0| > 1/e$.

Proof. Otherwise $f(B(0, 1)) \subset B(0, 1)$ and f would have a parabolic or attracting fixed point. \square

Lemma 3. *For each z in $J(f) \setminus P(f)$, there are arbitrarily small neighbourhoods U_z on which the first return map ϕ to U_z is expanding (that is, $|D\phi| > \gamma_z > 1$).*

Proof. This is part (iii) of Lemma 11 of [8]. \square

The following is a version of the classical Koebe distortion lemma.

Lemma 4. *For all large R , if g is any univalent map g on $B(0, R)$, the distortion of g on $B(0, 1)$ is bounded by $1 + 4/R$, that is,*

$$\sup_{y, z \in B(0, 1)} \left| \frac{Dg(y)}{Dg(z)} \right| < 1 + 4/R.$$

We shall denote by $\Delta > 1$ the modulus giving a distortion bound of 2, that is, the number such that for any univalent map g on $B(0, \Delta)$, the distortion of g on $B(0, 1)$ is bounded by 2.

Lemma 5. *Given any $\delta > 0$, there is a $\beta > 0$ such that, for any $z \in \mathbb{C}$ and $k \geq 0$, if $\text{dist}(f^k(z), P(f)) > \delta$ then $|Df^k(z)| > \beta$.*

Proof. Some neighbourhood W of z mapped biholomorphically onto $B(f^k(z), \delta/2)$ with a distortion bound depending only on δ . But f is not univalent on any ball of radius π , so W cannot strictly contain a ball of radius π . Combining these two facts, the derivative of f^k on W cannot be too small. \square

Lemma 6. *There is an $M > 3$ such that, for all $z \in \mathbb{C}$ and $k \geq 1$, if $|f^k(z)| \geq M$ then $|Df^k(z)| > 3$.*

Proof. Let $\delta > 0$ and let β be given by Lemma 5, Take $M > 3/\beta$ sufficiently large that $f(B(P(f), \delta)) \subset B(0, M)$. Then $|Df^{k-1}(z)| > \beta$. But $|Df^k(z)| = |f^k(z)| |Df^{k-1}(z)| \geq M\beta > 3$. \square

The same line of proof gives:

Lemma 7. *Given $M_1 > 0$ there is an $M_2 > 0$ such that, for all $z \in \mathbb{C}$ and $k \geq 1$, if $|f^k(z)| \geq M_2$ then $|Df^k(z)| > M_1 |f^k(z)|$.*

Lemma 8. *There exists $\delta > 0$ such that for all $z \in B(P(f), 3\delta)$, $|Df^{n_0}(z)| > \exp(\alpha/2)$.*

Proof. Continuity of the derivative. \square

Let $\delta < 1$ come from the preceding lemma and set $V := B(P(f), \delta)$.

Lemma 9. *There is some $\beta_1 > 0$ such that, for each $z \in \mathbb{C}$ and $k \geq 1$,*

$$|Df^k(z)| \geq \beta_1 \inf_{1 \leq j \leq k} |f^j(z)|.$$

Proof. It is easy to see that if $|Df^n(z)| \geq 1$, then

$$|Df^n(z)| \geq \inf_{1 \leq j \leq n} |f^j(z)|.$$

Let M be given by Lemma 6. Let $n \leq k$ be maximal such that $|f^n(z)| \geq M$. Then $|Df^n(z)| > 3$ and $|Df^k(z)| = |Df^n(z)||Df^{k-n}(f^n(z))| \geq |Df^{k-n}(f^n(z))|$, so it merely suffices to prove that

$$|Df^{k-n}(f^n(z))| \geq \beta_1 \inf_{k-n+1 \leq j \leq k} |f^j(z)|.$$

Rewriting, it suffices to prove the lemma under the assumption $|f^j(z)| < M$ for $j = 1, \dots, k$. Let W be the closure of $B(0, M) \setminus V$.

Let β be given by Lemma 5, and let $n \leq k$ be maximal such that $f^n(z) \in W$, if it exists, otherwise set $n = 0$. Then $|Df^n(z)| \geq \beta$. If $n = k$, we are done (provided $\beta_1 \leq \beta/M$). Now $f^{n+1}(z), \dots, f^k(z) \in V$, by definition of n . Since $|Df^{n_0}| > 1$ on V , it follows that $|Df^{k-n+1}(f^{n+1}(z))|$ is bounded below by the constant $\beta_2 := \min(\inf_{j < n_0} \inf_V |Df^j|, 1) > 0$. Then $|Df^k(z)| \geq \beta |f^{n+1}(z)| \beta_2$. Taking $\beta_1 := \beta \beta_2 / M$ works. \square

In the following lemma, we use exponential growth when one remains in a neighbourhood of $P(f)$, exponential growth when one remains in a bounded region disjoint from that neighbourhood, plus absolute growth if an iterate lands outside a large bounded region to give some sort of non-uniform hyperbolicity statement for Misiurewicz maps.

Lemma 10. *There are $c, N, N_1 > 0$ such that for each z there is a $j \leq N + c \log |f(z)|$ with $|Df^j(z)| > 3$ and $|Df^i(z)|, |f^i(z)| \leq N_1 + |f(z)|$ for $i = 1, \dots, j$.*

Proof. We can assume $|f(z)| \leq 3$, otherwise one can simply take $j = 1$. Let k be minimal such that $f^k(z) \notin V$. If $k \geq p := 2n_0 \lceil (2 - \log |f(z)|) / \alpha \rceil$, then

$$|Df^p(z)| \geq |f(z)| \exp(p\alpha/2n_0) \geq e^2$$

and we are done. Otherwise, by Lemma 5, there is a $\beta > 0$ such that $|Df^k(z)| > \beta$. It suffices to show that, there is an N such that for each $y \in f(V) \setminus V$ there is a $j \leq N$ with $|Df^j(y)| > 3/\beta$.

By Lemma 7, there is some $M > \text{diam}(V)$ such that, for any $z \in \mathbb{C}$, if $|f^n(z)| \geq M$ then $|Df^n(z)| > 3/\beta$. Thus we can set $N_1 := |\lambda_0|e^M$ and restrict our attention to those y which do not leave $B(0, M)$ for the first N iterates, for some large N to be defined. We can cover $W := B(0, M) \setminus V$ by a finite collection of balls $\{W_l\}_{l=1}^L$ on which the first return map is expanding, by Lemma 3, so there is a $\gamma > 1$ and each return map $\phi_l : W_l \rightarrow W_l$ has derivative greater than γ .

Let $p, r \in \mathbb{N}$ satisfy $\beta\gamma^p > 3/\beta$ and $\beta e^{r\alpha/2} |\lambda_0| e^{-M} > 3/\beta$. Set $N := pLrn_0$.

Consider the successive passages of y into W , at times k_0, k_1, \dots , say. By time k_{pL} , if such exists, there must be some W_l which is passed through at least p times. Then $|Df^{k_{pL}}(y)| > \beta\gamma^p > 3/\beta$ and if $k_{pL} \leq N$ we are done.

Otherwise, at some point the orbit must spend a long period, at least rn_0 long, in $B(0, M) \setminus W = V$, that is, there is some $a \geq 0$ such that $f^l(y) \in V$ for $l = a+1, \dots, a+rn_0 < N$ and such that $a = 0$ or $f^a(y) \in W$ with $|f^{a+1}(y)| \geq |\lambda_0|e^{-M}$. But by definition of V ,

$$|Df^{rn_0}(f^{a+1}(y))| \geq \exp(r\alpha/2).$$

The choice of r entails $\beta |Df^{rn_0}(f^{a+1}(y))| |f^{a+1}(y)| > 3/\beta$, so $|Df^{a+1+rn_0}(y)| > 3/\beta$. \square

Lemma 11. *If $f^k(z) \in V$ for all $0 \leq k \leq n_0p$ then there is a neighbourhood of z , contained in $B(z, 2\delta)$, mapped biholomorphically by f^{n_0p} onto $B(f^{n_0p}(z), 2\delta)$.*

Proof. By induction. \square

Recall that $\Delta > 1$ is the constant giving a Koebe distortion bound of 2.

Lemma 12. *Given $\varepsilon > 0$ there is a $\delta_0 > 0$ such that the following holds. If $f^k(z) \in V$ for all $0 \leq k \leq p$ then there is a neighbourhood V_k of z , contained in $B(z, \varepsilon/|Df^k(z)|)$, mapped biholomorphically by f^p onto $B(f^p(z), \Delta\delta_0)$.*

Proof. Use the preceding lemma to reduce it to fewer than n_0 iterates, and use that V is bounded, and bounded distortion of f on V . \square

Lemma 13. *There exists $\delta_0 > 0$ such that if $z \in \mathbb{C}$ and $f^j(z) \in V$ for $j = 1, \dots, k$, and $|Df^k(z)| > 1$, then there is a neighbourhood U of z mapped biholomorphically by f^k onto $B(f^k(z), \Delta\delta_0)$ with $U \subset B(z, \delta)$.*

Proof. By hypothesis, $|Df^k(z)| = |f(z)| |Df^{k-1}(z)| > 1$, so letting $\varepsilon < \delta/10$ and taking δ_0 from the preceding lemma, there is a neighbourhood of $f(z)$ contained in $B(f(z), \varepsilon|f(z)|)$ mapped biholomorphically onto $B(f^k(z), \Delta\delta_0)$. By properties of the exponential map, the relevant pullback of $B(f(z), \varepsilon|f(z)|)$ is contained in $B(z, \delta)$. \square

Lemma 14. *Let $\delta_0 > 0$ be given by Lemma 13. Let $z \in \mathbb{C}$ and suppose $|Df^k(z)| > |Df^j(z)|$ for all $j = 0, \dots, k-1$. Then there is a neighbourhood of z mapped biholomorphically by f^k onto $B(f^k(z), \Delta\delta_0)$.*

Proof. Let $j \leq k$ be maximal such that $f^j(z) \notin V$. If such a j does not exist, we can simply apply Lemma 12. Otherwise, applying Lemma 13, there is a neighbourhood U of $f^j(z)$ mapped by f^{k-j} biholomorphically onto $B(f^k(z), \Delta\delta_0)$, and $U \subset B(f^j(z), \delta)$, so $U \cap P(f) = \emptyset$. Therefore there is a neighbourhood of z mapped biholomorphically onto U . \square

3. MEASURE ESTIMATES

Denote by $\mathcal{R}(x)$ the right half-plane $\{z \in \mathbb{C} : \Re(z) > x\}$ and denote by $\mathcal{L}(x)$ the left half-plane $\mathbb{C} \setminus \mathcal{R}(x)$. Denote by \mathcal{Q} the collection of squares of the form

$$\{z : 2k\pi \leq \Re(z) < (2k+2)\pi; 2j\pi \leq \Im(z) < (2j+2)\pi\},$$

for $j, k \in \mathbb{Z}$. Each square has diameter $2\sqrt{2}\pi$.

Let $\delta_0 > 0$ be given by Lemma 14. Let N_1 be given by Lemma 10. By Lemma 7, there is an $M > 100$ such that, if $|f^k(z)| > M$ then $|Df^k(z)| > |f^k(z)|8\pi/\delta_0$. We can suppose that $M > |\lambda_0|e^{N_1+2\Delta}$.

Lemma 15. *For any $Q \in \mathcal{Q}$ with $Q \subset \mathcal{R}(M)$, and any U, n such that f^n maps U biholomorphically onto Q , the distortion of f^n on U is bounded by 2.*

Lemma 16. *There is a finite collection of sets U_1, \dots, U_p with corresponding numbers n_j such that f^{n_j} maps U_j biholomorphically onto a square from \mathcal{Q} in $\mathcal{R}(M)$, and such that for each y with $|y| \leq 2M$, $B(y, \delta_0)$ contains some U_j .*

Proof. By transitivity of f , there is finite set Z such that $\text{dist}(y, Z) < \delta_0/2$ for all y with $|y| \leq 2M$, and such that for each $z \in Z$, there is an n such that $f^n(z) \in \mathcal{R}(M)$. Now if z gets mapped by some f^n inside a square $Q \in \mathcal{Q}$ with $Q \subset \mathcal{R}(M)$, then there is a neighbourhood U of z of diameter less than $2\sqrt{2}\pi\delta_0/8\pi = \delta_0/2\sqrt{2}$ which gets mapped by f^n biholomorphically onto Q . If $|y - z| < \delta_0/2$, $U \subset B(y, \delta_0)$. The result follows. \square

Lemma 17. *There is a collection of sets $\{U_i\}_{i \geq 0}$ and a constant $K > 1$ such that each U_i is mapped by some f^n onto a square $Q \in \mathcal{Q}$ with $Q \subset \mathcal{R}(M)$ with derivative bounded by K and distortion bounded by 2, and such that, if $\Re(y) \leq M$, then $B(y, \delta_0)$ contains an element from $\{U_i\}_{i \geq 0}$.*

Proof. Taking translates by multiples of $2\pi i$ of the collection of sets U_1, \dots, U_p from Lemma 16 deals with the points with $-M \leq \Re(y) \leq M$.

If $\Re(y) < -M$, by Lemma 10 there is a least $j \geq 1$ with $3 < |Df^j(y)|$ and for this j , $|f^j(y)|, |Df^j(y)| < N_1$. By Lemma 14, there is a neighbourhood of y mapped biholomorphically onto $B(f^j(y), \Delta\delta_0)$ with a corresponding neighbourhood mapped V onto $B(f^j(y), \delta_0)$ with distortion bounded by 2 (by choice of Δ). On V we deduce $1 < 3/2 < |Df^j(y)| < 2N_1$. Now $B(f^j(y), \delta_0)$ contains one of the sets U_1, \dots, U_p , and we can pull this set back with uniformly bounded derivative and distortion.

The tighter distortion bound comes from Lemma 15. \square

Lemma 18. *Let $Q \in \mathcal{Q}$ satisfy $Q \subset \mathcal{R}(y) \setminus \mathcal{R}(y + 7)$ for some $y \geq M$. Then there is a subset of Q mapped biholomorphically onto a square $Q' \in \mathcal{Q}$ satisfying $Q' \subset \mathcal{R}(|\lambda_0|e^y/2) \setminus \mathcal{R}(|\lambda_0|e^y e^7)$.*

Proof. Let Z denote the cone of positive linear combinations of $1 + i$ and $1 - i$. Then one quarter of any square of \mathcal{Q} gets mapped injectively into Z . We have $f(Q) \cap Z \subset \mathcal{R}(|\lambda_0|e^y/\sqrt{2})$, and $f(Q) \cap \mathcal{R}(|\lambda_0|e^{y+7}) = \emptyset$. Only a small proportion of squares from \mathcal{Q} in $f(Q) \cap Z$ intersect $f(\partial Q)$, so we can pull back one of the other squares to get the required subset. \square

Lemma 19. *Let $x > M$ and suppose $Q \in \mathcal{Q}$ satisfies $Q \subset \mathcal{R}(M)$. For some $z \in Q$ and some $k \geq 0$, the ball $B(z, 1/x^3) \subset Q$ is mapped by f^k univalently into $\mathcal{R}(x)$.*

Proof. Suppose $Q \subset \mathcal{R}(y) \setminus \mathcal{R}(y + 7)$. By repeatedly applying Lemma 18, we can construct an increasing sequence of numbers $y = y_0 < y_1 < y_2 < \dots$ and a decreasing sequence of sets $Q = V_0 \supset V_1 \supset \dots$ such that the following holds. For each $k \geq 0$,

- $f^k(V_k) \in \mathcal{Q}$;
- $e^{y_k/2} < y_{k+1} < e^7 |\lambda_0| e^{y_k}$;
- $f^k(V_k) \subset \mathcal{R}(y_k) \setminus \mathcal{R}(y_k + 7)$;

The derivative $|Df^k|$ on V_k is, for each k , bounded by y_k^2 . The distortion of f^k on V_k is bounded by 2 (by Lemma 15).

Let $k \geq 0$ be minimal such that $y_k \geq x$. If $f^k(V_{k-1}) \subset \mathcal{L}(2ex)$ then $y_k \leq 2ex + 7$ and $|Df^k|$ on V_k is bounded by $(2ex + 7)^2$. Therefore V_k easily contains a ball of radius $1/x^3$. Otherwise, $f^k(V_{k-1})$ is a geometric annulus centred on zero and intersecting $\mathcal{R}(2ex)$, and the square $f^{k-1}(V_{k-1})$ contains a ball of radius $2\pi/16$ mapped by f into $\mathcal{R}(x)$, as is easy to check. The derivative of f^{k-1} on V_{k-1} is

bounded by $y_{k-1}^2 < x^2$, so pulling back the ball we get a set containing a ball of radius $1/x^3$ once again, as required. \square

Lemma 20. *There is a constant $\gamma > 0$ such that if $x > M$ the following holds.*

If $Q \in \mathcal{Q}$, there is a ball of radius γ/x^3 inside Q which gets mapped univalently by some f^n into $\mathcal{R}(x)$ with distortion bounded by 2.

If $\Re(y) < M$, then there is a ball of radius γ/x^3 inside $B(y, \delta_0)$ which gets mapped univalently by f^n , for some $n \geq 0$, into $\mathcal{R}(x)$ with distortion bounded by 2.

Proof. This follows from Lemmas 17, 19. \square

The preceding lemma says that a certain proportion of everything at the large scale gets mapped far out to the right. The next lemma deduces the same, but at small scales.

Lemma 21. *There are constants $\kappa > 0, M_0 \geq M$ such that the following holds. Given $r \in (0, 1)$, $x \geq M_0$ and $z \in \mathbb{C}$, there is a finite collection of pairwise disjoint balls $B_i \subset B(z, r)$, each of radius $> e^{-2x}r$, such that*

- $m(\bigcup_i B_i)/m(B(z, r)) > \kappa/x^6$;
- for each B_i there is an $n_i \geq 0$ with $f^{n_i}(B_i) \subset \mathcal{R}(x)$;
- $|Df^{n_i}|_{B_i} < e^{3x}/r$.

Proof. Let n be minimal such that $|Df^n(z)| > 10/r$. If there is some minimal $k < n$ with $f^k(z) \in \mathcal{R}(x)$, we can just pull back $B(f^k(z), 1)$ to get a set containing $B(z, r/5)$, using the derivative estimate and a distortion bound of 2. Some large sector of $B(z, r/5)$ gets mapped to $\mathcal{R}(\Re(f^k(z)))$ and the lemma follows easily.

Otherwise, $f^k(z) \notin \mathcal{R}(x)$ for $k = 0, 1, \dots, n-1$, and $|Df^n(z)| \leq |\lambda_0|e^x 10/r$.

If $|f^n(z)| < M$, then f^n maps some neighbourhood W of z univalently onto $B(f^n(z), \delta_0)$ with distortion bounded by 2 by Lemma 14. By Lemma 20, there is a $\gamma > 0$ such that a ball of radius γ/x^3 in $B(f^n(z), \delta_0)$ that gets mapped with distortion bounded by 2 into $\mathcal{R}(x)$. As $|Df^n| < 2|f^n(z)|10/r < 20M/r$ on W , pulling back this ball gives a subset of W containing a ball of radius $(\gamma/x^3)r/20M$, as required.

If $|f^n(z)| \geq M$, then $|f^{n-1}(z)|$ is large, so some neighbourhood W of z gets mapped with distortion bounded by 2 onto $B(f^{n-1}(z), 1)$, and $W \supset B(z, r/20)$. On $B(f^{n-1}(z), 1)$, f is univalent with distortion bounded by e^2 . Thus on W , the distortion of f^n is bounded by $4e^2$. Then the squares $\{Q \in \mathcal{Q} : Q \subset f^n(W)\}$ fill most of $f^n(W)$. In particular, there is a collection of pairwise disjoint subsets $W_i \subset W$, each mapped by f^n onto an element Q_i of \mathcal{Q} with $m(\bigcup_i W_i)/m(W) > 1/2$, say. One can apply Lemma 20 on each Q_i and pull back to get a subset $V_i \subset W_i$ mapped by f^n onto a ball of radius γ/x^3 , and some $l \geq n$ such that f^l maps V_i with distortion bounded by 2 into $\mathcal{R}(x)$. The distortion bound implies V_i contains a ball B_i of radius $\text{diam}(V_i)/4$, so $m(B_i)/m(V_i) > 1/16$. We can assume l is minimal such that $f^l(V_i) \subset \mathcal{R}(x)$. The $f^{l-1}(V_i) \not\subset \mathcal{R}(x+1)$, and $f^{l-1}(V_i)$ does not contain a ball of radius π . The bound on $|Df^n_W|$ implies B_i has radius $\geq (\gamma/x^3)r/20|\lambda_0|e^x$. Therefore

$$|Df_{V_i}^{l-1}| < 20\pi|\lambda_0|e^x x^3/r\gamma.$$

Thus $|Df_{V_i}^l| < 40\pi|\lambda_0|^2 e^{2x+1} x^3/r\gamma < e^{3x}/r$, if x is large enough.

We note to finish that $m(V_i)/m(W_i) > \gamma^2/4x^6$ for each i . Combining this with the uniform estimates for $m(B_i)/m(V_i)$, $m(\bigcup_i W_i)/m(W)$ and $m(W)/m(B(z, r))$, we conclude $m(\bigcup_i B_i) > \kappa/x^6 m(B(z, r))$ for some $\kappa > 0$ independent of x . \square

We call a square D *dyadic* if $2\pi 2^k D$ is an element of \mathcal{Q} for some integer $k \geq 1$; 2^{-k} is then called the *scale* of D . Since each ball contains a square of comparable size, and vice versa, we the previous lemma also holds for dyadic squares, with perhaps a slightly smaller scale (which we estimate crudely).

Lemma 22. *There are constants $\kappa > 0$, $M_0 \geq M$ such that the following holds. Let $k \geq 3$. Let $x \geq M_0$ and let $D \subset \mathcal{R}(-e^x)$ be a dyadic square of scale 2^{-k} . Then there is a finite collection of pairwise disjoint dyadic squares $D_i \subset D$, each of scale $> e^{-3x} 2^{-k}$, such that*

- $m(\bigcup_i D_i)/m(D) > \kappa/x^6$;
- for each D_i there is an $n_i \geq 0$ with $f^{n_i}(D_i) \subset \mathcal{R}(x)$;
- f^{n_i} is univalent on $B(z, \Delta \text{diam}(D_i))$ for all $z \in D_i$;
- $|Df_{|D_i}^{n_i}| < e^{3x} 2^k$.

If at all scales, a certain proportion gets mapped far out to the right, then almost every point does. The next lemma gives bounds on the time needed for a *large* proportion of points to get mapped far out to the right, together with a bound on the corresponding derivatives.

Lemma 23. *Let S be a bounded neighbourhood of $P(f)$. There is a constant M_0 such that the following holds. Let $x > M_0$. Let S_* denote the set of points z such that the first entry to $\mathcal{R}(x)$ happens at time $n(z)$ with*

- $|Df^{n(z)}(z)| < e^{x^9}$;
- $n(z) \leq e^{2x}$.

Then $m(S \setminus S_)/m(S) \leq 1/x$.*

Proof. Let κ, M_0 come from Lemma 22. We can cover S with a finite number of dyadic squares of scale 2^{-3} , each contained in $B(S, 1)$, and with total area A , say. If $M' > M_0$ is sufficiently large, $x > M'$ and $p = x^7$, then

$$(1 - \kappa/x^6)^p A < 1/x.$$

At least a proportion κ/x^6 of each of these dyadic squares is covered by dyadic squares of scale $\geq 2^{-3}e^{-3x}$ given by Lemma 22. The remainder, less than $(1 - \kappa/x^6)$, can be covered by other dyadic squares of size $\geq 2^{-3}e^{-3x}$ and we can apply Lemma 22 to each of these squares. Proceeding inductively, after p such applications, we end up with a collection \mathcal{D} of dyadic squares such that

$$m(S \setminus \bigcup_{D \in \mathcal{D}} D) \leq (1 - \kappa/x^6)^p A < 1/x$$

and such that each $D \in \mathcal{D}$ satisfies

- the scale of D is $\geq 2^{-3}(e^{-3x})^p$;
- there is an $n_D \geq 0$, with $f^{n_D}(D) \subset \mathcal{R}(x)$;
- f^{n_D} is univalent on $B(z, \Delta \text{diam}(D))$ for all $z \in D$;
- $|Df_{|D}^{n_D}| < (e^{3x})^{p+1}$.

We wish to show that S_* contains $\bigcup_{D \in \mathcal{D}} D$. For a point $y \in D \in \mathcal{D}$, n_D is not necessarily the first entry time $n(y)$ to $\mathcal{R}(x)$, but for all $j < n$, $3|Df^j(y)| < |Df^{n_D}(y)|$, so $|Df^{n(y)}(y)| < (e^{3x})^{p+1} < e^{x^9}$.

It remains to show that n_D is not too large. While n_D may not be the first entry time for all points $y \in D$, it can be assumed that it is for some point $z \in D$. Now f^j on $B(z, \text{diam}(D))$ is univalent with distortion bounded by 2 for all $j \leq n_D$, by choice of Δ , so $f^j(B(z, \text{diam}(D)))$ cannot contain a ball of radius π for any $j < n_D$ and thus has diameter bounded by 4π . In particular, it does not intersect $\mathcal{R}(x+4\pi)$. Thus for $1 \leq j \leq n_D$, $f^j(D) \subset B(0, |\lambda_0|e^{x+4\pi})$.

By Lemma 10, inside the region $B(0, |\lambda_0|e^{x+4\pi})$ the derivative multiplies by at least 3 at least every $C_0 e^x$ steps for some $C_0 > 0$. Thus,

$$3^{n_D/C_0 e^x} < |Df_D^{n_D}|,$$

so taking logs and using the estimate for the derivative,

$$n_D/C_0 e^x < 3x(p+1),$$

$$n_D < C_0(p+1)(3x)e^x < e^{2x}.$$

We reset $M_0 := M'$. □

A key claim in the next lemma is that for many points, the first entry to $\mathcal{L}(-e^x)$ actually lands in $\mathcal{L}(-e^{x+\sqrt{x}})$. This added distance will be crucial later on to kill the derivative in the subsequent iterate.

Lemma 24. *Let $S := B(P(f), 1)$. There is a constant M_1 such that the following holds. Let $x > M_1$. Let S_0 denote the set of points $z \in B(P(f), 1 - x^{-1/4})$ such that the first entry to $\mathcal{L}(-e^x)$ happens at time $n(z)$ with*

- $f^{n(z)}(z) \in \mathcal{L}(-e^{x+\sqrt{x}})$
- $e^x < |Df^{n(z)}(z)| < e^{x^9} |\Re(f^{n(z)}(z))|^3$;
- $\inf_{j+k \leq n(z)} |Df^j(f^k(z))| > \exp(-e^{x+1})$;
- $n(z) \leq e^{3x}$;
- $f^{n(z)}$ maps a neighbourhood of z onto the element of \mathcal{Q} containing $f^{n(z)}(z)$ with distortion bounded by $1/x$.

Then $m(S \setminus S_0) \leq 1/\log x$.

Proof. The length of ∂S is bounded, so there is some $C > 0$ such that, setting $S_x := B(P(f), 1 - 2x^{-1/4})$, we have $m(S_x)/m(S) > 1 - Cx^{-1/4}$ for all $x > 1$.

Let S_* be given by Lemma 23 for some sufficiently large x . Set $S' := S_* \cap S_x$, so $m(S')/m(S) > 1 - 1/x - Cx^{-1/4}$. Let $z \in S'$ and let n be the associated number $n(z)$ given by the lemma, so $|Df^n(z)| < e^{x^9}$.

Suppose first that $\Re(f^n(z)) < x + x^{3/4}$. Let T denote the partial strip

$$\{z : x \leq \Re(z) < x + x^{3/4}; 2j\pi \leq \Im(z) < (2j+2)\pi\}$$

containing $f^n(z)$, for the relevant integer j . By Lemma 7, the neighbourhood W_* of z mapped univalently by f^n onto T has diameter less than $x^{3/4}/x = x^{-1/4}$, so $W_* \subset B(P(f), 1 - x^{-1/4})$. Let $T_+ := T \cap \mathcal{R}(x + 2\sqrt{x})$, and set $W := \{z \in W : f^n(z) \in T_+\}$. Then

$$m(W)/m(W_*) \geq 1 - 4\sqrt{x}/x^{3/4} = 1 - 4x^{-1/4},$$

using a distortion bound of 2.

If $\Re(f^n(z)) \geq x + x^{3/4}$, let T denote the partial strip

$$\{z : x + k \leq \Re(f^n(z)) < x + k + 1; 2j\pi \leq \Im(z) < (2j+2)\pi\}$$

containing $f^n(z)$, for the relevant integers k, j . As before, by Lemma 7, the neighbourhood W of z mapped univalently by f^n onto T has diameter less than $1/x$, and $f|_W$ has distortion bounded by 2. Thus $W \subset B(P(f), 1-1/x) \subset B(P(f), 1-x^{-1/4})$.

Since we can do this for each $z \in S'$, there is a collection \mathcal{W} of pairwise disjoint subsets $W \subset B(P(f), 1-x^{-1/4})$ with $m(\bigcup_{W \in \mathcal{W}} W)/m(S') \geq 1-4x^{-1/4}$. Moreover each W is mapped univalently onto a partial strip $T \subset \mathcal{R}(x+2\sqrt{x})$. Standard arguments show that there is a set $T_W \subset T$ of measure $(1-1/x)m(T)$ such that for each $y \in T_W$, for some $0 \leq k \leq x$,

- $\Re(f^{k+1}) < -|f^k(y)|^2 < -e^{x+\sqrt{x}}$;
- $|f^{k+1}(y)| < |\Re(f^{k+1}(y))|^2$,

and, for $0 \leq j \leq k$, $\Re(f^{j+1}(y)) > |f^j(y)|^2$. Pulling back the T_W to the W (with distortion bounded by 2) and taking a union over $W \in \mathcal{W}$, we get a set S_0 with $m(S_0)/m(S') \geq 1-2/x-4x^{-1/4}$ and the desired properties: The lower derivative bounds follow from Lemmas 7 and 9 (note that $|f^j(z)| \geq |\lambda_0| \exp(-e^x)$ for $j \leq n$). The distortion bound is given by Lemma 4.

Thus $m(S_0) \geq (1-2/x-4x^{-1/4}-1/x-Cx^{-1/4})m(S)$. For large x , this expression is greater than $1-1/\log x$. \square

4. PARAMETRIC ESTIMATES

We denote by \log the branch of logarithm sending a neighbourhood of 1 in \mathbb{C} to a neighbourhood of 0.

Lemma 25. *Let $\gamma, \beta \in (0, 1/10)$. Let $\lambda_1, \lambda_2 \in \mathbb{C} \setminus \{0\}$ with $|\log(\lambda_1/\lambda_2)| < \beta$, and let $g_j : z \mapsto \lambda_j e^z$ for $j = 1, 2$. Suppose $y, z \in \mathbb{C}$ and $g_1(z') = z$. If $|(y-z)/z| < \gamma$, then there is a point y' close to z with $g_2(y') = y$ with*

$$|y' - z' - (y - z)/z| < 3\gamma^2 + \beta.$$

Proof. Dividing gives $y/z = e^{y'-z'} \lambda_2/\lambda_1$ for any preimage y' of y , in particular the closest one to z' . Taking logs, we have $\log(y/z) = \log(\lambda_2/\lambda_1) + y' - z'$. Then $y' - z' = (y-z)/z + \log(\lambda_2/\lambda_1) + \log(y/z) - (y-z)/z$. Writing $y/z = 1 + (y-z)/z$ and expanding \log gives $|\log(y/z) - (y-z)/z| < 3\gamma^2$. The result follows. \square

Lemma 26. *Let $x > 10$. Let $\lambda_1, \lambda_2 \in \mathbb{C} \setminus \{0\}$ with $\beta := |\log(\lambda_1/\lambda_2)| < \exp(-5e^{x+1})$, and let $g_j : z \mapsto \lambda_j e^z$ for $j = 1, 2$. Suppose $n > 0, z = z_n \in \mathbb{C}$, and $g_1(z_{j+1}) = z_j$ for $j = 0, 1, \dots, n-1$.*

- $\inf_{j+k \leq n} |Dg_1^j(g_1^k(z))| > \exp(-e^{x+1})$;
- $n \leq e^{3x}$.

Then there is a y_n with $|g_2^j(y_n) - g_1^j(z_n)| < \beta \exp(e^{x+2})$ for all $j \leq n$ and $g_2^n(y_n) = g_1^n(z)$. Moreover $|\log Dg_2^n(y_n)/Dg_1^n(z_n)| < \exp(-e^x)$.

Proof. Let us assume we have constructed y_j for $j < k$, and $1 - y_j/z_j$ is small. Set $\alpha_j := \log(\lambda_1/\lambda_2) + \log(y_j/z_j) - (y_j - z_j)/z_j$. Then set

$$y_k = z_k + (y_{k-1} - z_{k-1})/z_{k-1} + \alpha_{k-1},$$

so $g_2(y_k) = y_{k-1}$, and define α_k appropriately. It follows that

$$y_k - z_k = \sum_{j=0}^{k-1} \frac{\alpha_j}{Dg^{k-j}(z_k)}.$$

Thus $|y_k - z_k| \leq e^{3x} \exp(e^{x+1}) \max_{j < k} |\alpha_j|$. For $k < n$, $|z_k| > \exp(-e^{x+1})$ (by the derivative estimate), so

$$|y_k - z_k|/|z_k| \leq e^{3x} \exp(2e^{x+1}) \max_{j < k} |\alpha_j|.$$

Applying Lemma 25,

$$|\alpha_k| < \beta + 3e^{6x} \exp(4e^{x+1}) \max_{j < k} |\alpha_j|^2 < \beta + \beta^{-1} \max_{j < k} |\alpha_j|^2/4.$$

Now $|\alpha_0| = \beta$, so by induction it follows that $|\alpha_j| < 2\beta$ for all $j \leq k$. Hence $|y_k - z_k| < 2\beta e^{3x} \exp(e^{x+1}) < \beta \exp(e^{x+2})$. The existence of y_n follows by induction.

For the derivative estimate, note that

$$\left| \log \frac{Dg_2^n(y_n)}{Dg_1^n(z_n)} \right| = \left| \sum_{j=0}^{n-1} \log y_j/z_j \right| \leq \sum_{j=0}^{n-1} 2 \left| \frac{y_j - z_j}{z_j} \right| \leq e^{3x} 4\beta e^{3x} \exp(e^{x+1}) < \exp(-e^x).$$

□

The preceding lemma allows us to show that some sets which get mapped eventually onto a square far out to the left do not move very fast as the parameter λ varies, so if λ does not vary much, the intersection remains large. Later on we will show that for relatively large sets of parameters, the orbit of 0 under f_λ lands in one of these intersections.

Given a function $R : \mathbb{C}^2 \rightarrow \mathbb{C}$, for $j = 1, 2$ we let $D_j R(z_1, z_2)$ denote the partial derivative of R with respect to the j^{th} variable, evaluated at the point (z_1, z_2) .

Lemma 27. *Let $C, \nu_0 > 0$. There is an $M_2 > 0$ such that for $x > M_2$, the following holds. Suppose $A \subset B(P(f), 1)$ satisfies $m(A) > \nu_0^2$ and $m(B(\partial A, \nu))/m(A) < C\nu$ for all $\nu \in (0, \nu_0)$. Let $B := \{\lambda : |\log(\lambda_0/\lambda)| < \exp(-5e^{x+1})\}$. There is a collection $\{U_l\}_{l=1}^L$ of pairwise disjoint subsets of A and numbers n_l , together with a map $R : \bigcup_l U_l \times B \rightarrow A \setminus B(\partial A, e^{-x})$ such that*

- $R(z, \lambda_0) = z$;
- on each $U_l \times B$, R is holomorphic, $|\log D_1 R| < \exp(-e^x)$ and $|D_2 R| < \exp(e^{x+3})$;
- $m(\bigcup_l U_l)/m(A) \geq 1/\log \log x$;
- for each λ , the sets $R(U_l, \lambda)$ for $l = 1, \dots, L$ are pairwise disjoint;
- $f_\lambda^{n_l}(R(U_l, \lambda)) \subset \mathcal{L}(-e^{x+\sqrt{x}} + 2\pi)$;
- $|Df_\lambda^{n_l}|_{|R(U_l, \lambda)} < e^{x^9} \inf_{z \in R(U_l, \lambda)} |\Re(f^{n_l}(z))|^4$;

Proof. Let $S := B(P(f), 1)$ and let M_1 be given by Lemma 24. Let $x > M_1$. Let S_0 be given by Lemma 24, so $m(A \setminus S_0) \leq m(S \setminus S_0) \leq 1/\log x$. Then we can cover S_0 by neighbourhoods U_z of points $z \in S_0$ each mapped by $f^{n(z)}$ biholomorphically onto the element Q_z of \mathcal{Q} containing $f^{n(z)}(z) \in \mathcal{L}(-e^{x+\sqrt{x}})$. Clearly $Q_z \subset \mathcal{L}(-e^{x+\sqrt{x}} + 2\pi)$. Since $n(z)$ is also the first entry time of z to $\mathcal{L}(-e^x)$, it follows that if $z' \in S_0$ and $Q_z \cap Q_{z'} \neq \emptyset$ then $Q_z = Q_{z'}$. Thus the neighbourhoods U_z form a finite (since $n(z)$ is bounded), pairwise disjoint collection which we can write as $\{U_l\}_{l=1}^L$, setting $n_l := n(z)$ for some (or any) $z \in U_l$. The estimate on the diameter of U_l comes from Lemma 7, for example.

Now the distortion of f^{n_l} is bounded by 2 on each U_l by Lemma 15, so

$$|Df^{n_l}|_{|U_l} < 2e^{x^9} \sup_{z \in U_l} |\Re(f^{n_l}(z))|^3 < e^{x^9} \inf_{z \in U_l} |\Re(f^{n_l}(z))|^4/2.$$

We can therefore use Lemma 26 to study the holomorphic map $R_l : U_l \times B \rightarrow \mathbb{C}$, where $R_l(z, \lambda_0) = z$ and

$$R_l : (z, \lambda) \mapsto f_\lambda^{-n_l} \circ f^{n_l}(z).$$

By Lemma 26, $|\log D_1 R_l| < \exp(-e^x)$. This gives the bound for $|Df_\lambda^{n_l}|$ on $R_l(U_l, \lambda)$. Meanwhile, the lemma also gives that for each $z \in U_l$ and $\lambda_1 \in B$ we have $|R_l(z, \lambda_1) - R_l(z, \lambda)| < |\log(\lambda/\lambda_1)| \exp(e^{x+2})$. It follows easily from this that $|D_2 R_l(z, \lambda_1)|$ is bounded by $4 \exp(e^{x+2})/|\lambda_1| < \exp(e^{x+3})$, and this for any $\lambda_1 \in B$. Again, since n_l is the first entry time of any point of $R_l(U_l, \lambda)$ (under iteration by f_λ) to $\mathcal{L}(-e^x - 1)$, the sets $R_l(U_l, \lambda)$, $1 \leq l \leq L$, are pairwise disjoint. Define R as the map whose restriction to each U_l is R_l .

It remains to show that a subcollection of the sets U_l has large enough measure. We have that each U_l from Lemma 27 has diameter less than e^{-x} . By the bound on $|D_2 R|$, we have $\max_{\lambda \in B} |R(z, \lambda) - R(z, \lambda_0)| < \exp(e^{x+3}) \text{diam}(B) < e^{-x}$, so the combined measure of those U_l for which $R(U_l, \lambda) \cap B(\partial A, e^{-x}) \neq \emptyset$ for some $\lambda \in B$ is less than $3Ce^{-x}$. Thus letting \mathcal{U} denote the collection of U_l for which $R(U_l, \lambda) \subset A \setminus B(\partial A, e^{-x})$ for all $\lambda \in B$, we have

$$m(A \setminus \bigcup_{U_l \in \mathcal{U}} U_l) < 1/\log x + 3Ce^{-x} < 2/\log x$$

for large x . Meanwhile, $m(A) > \nu_0^2$, so we deduce

$$m(\bigcup_{U_l \in \mathcal{U}} U_l)/m(A) \geq 1 - 2/m(A) \log x > 1 - 2/\nu_0^2 \log x > 1 - 1/\log \log x,$$

again for all large x . □

5. PROOF OF MAIN THEOREM

Let us write $\xi_n(\lambda) := f_\lambda^n(0)$. Let $A(y; a_1, a_2)$ denote the annulus centred on $y \in \mathbb{C}$ with inner and outer radii of lengths a_1, a_2 . Given $k \geq 2$, the k rays leaving y with angles $2j\pi/k$ for $j \leq k$ divide $A(y; a_1, a_2)$ into k congruent pieces which we will call k -sectors of $A(y; a_1, a_2)$.

Let δ_0 be given by Lemma 13. From Lemma 4.1 and the lines preceding it in [2], there are $K, C_1, \delta' > 0$ with $\delta' < \delta_0/2$ such that the following holds. Let $\varepsilon > 0$. There are $\gamma < 1$ and $r_0, n, C, \nu_0 > 0$ such that the following holds. Let $r \in (0, r_0)$.

- $\xi_n : A(\lambda_0; \gamma r, r) \rightarrow B(P(f), \delta_0)$ has distortion bounded by $1 + \varepsilon$;
- $|D\xi_n| > \delta'/C_1 r$;
- $|Df_\lambda^n(0)| < C_1/r^K$;
- the map ξ_n is univalent on each $2K$ -sector A_j , $1 \leq j \leq 2K$, of $A(\lambda_0; \gamma r, r)$;
- $\delta'/C_1 < \text{diam}(\xi_n(A_j)) < \delta'$;
- for all $\nu \in (0, \nu_0)$,

$$m(B(\partial \xi_n(A_j), \nu))/m(\xi_n(A_j)) < C\nu.$$

Note that the last statement follows from the preceding one and bounded distortion of ξ_n .

Let M_2 be given by Lemma 27 for C and ν_0 . Let

$$r_1 := \min(r_0, \exp(-5e^{M_2+3}), (\delta'/C_1)^2).$$

Let $r < r_1$ and let x satisfy $r = \exp(-5e^{x+3})$, so $x > M_2$. Fix some $2K$ -sector A_j of $A(\lambda_0; \gamma r, r)$ and set $A := \xi_n(A_j)$.

For $\lambda \in B(\lambda_0, r)$, we have $|\log(\lambda_0/\lambda)| < \exp(-5e^{x+1})$. Thus we can apply Lemma 27. Let $\{U_l\}_{l=1}^L$ and the map R be given by the lemma. Fix l for now, and let $z \in U_l$. Let

$$W_z := \{R(z, \lambda) : \lambda \in B\} \subset A \setminus B(\partial A, e^{-x}).$$

As $|D_2 R| \ll |D\xi_n|$, the map $y \mapsto R(z, \xi_n^{-1}(y))$ is a strict contraction on W_z and it has a unique fixed point $y_z \in \overline{W_z} \subset A$. Let $\Lambda(z) := \xi_n^{-1}(y_z)$, so $\xi_n(\Lambda(z)) = R(z, \Lambda(z))$. Now $D\xi_n - D_2 R \neq 0$, so we can apply the implicit function theorem to deduce that $z \mapsto \Lambda(z)$ is holomorphic on each U_l . Suppose $\Lambda(z) = \Lambda(z_1)$. From Lemma 27, for each λ , the sets $R(U_l, \lambda)$ are pairwise disjoint, so z and z_1 must be in the same U_l . But on each $U_l \times \{\lambda\}$, R is a homeomorphism, so $z = z_1$. Thus $\Lambda(z)$ is injective on $U := \bigcup_{l=1}^L U_l$.

Taking derivative of $\xi_n(\Lambda(z)) = R(z, \Lambda(z))$ with respect to z ,

$$D\xi_n(\Lambda(z))D\Lambda(z) = D_1 R(z, \Lambda(z)) + D_2 R(z, \Lambda(z))D\Lambda(z),$$

so

$$D\xi_n(\Lambda(z))D\Lambda(z) = D_1 R(z, \Lambda(z)) \left(1 - \frac{D_2 R(z, \Lambda(z))}{D\xi_n(z)}\right)^{-1}.$$

Now

$$\left| \frac{D_2 R(z, \Lambda(z))}{D\xi_n(\Lambda(z))} \right| < \exp(e^{x+3})rC_1/\delta' < \exp(-e^x),$$

and $|D_1(R(z, \Lambda(z)))| < \exp(-e^x)$. Thus $|D\Lambda(z)D\xi_n(\Lambda(z)) - 1| < e^{-x}$, say. In particular,

$$(1 - e^{-x})^2 \int_U |D\xi_n^{-1}|^2 dm \leq \int_U |D\Lambda|^2 dm = m(\Lambda(U)).$$

But the distortion bound on ξ_n gives

$$\int_U |D\xi_n^{-1}|^2 dm / m(A_j) \geq (1 - \varepsilon)m(U)/m(A) \geq (1 - \varepsilon)(1 - 1/\log \log x).$$

Thus $m(\Lambda(U))/m(A_j) \geq (1 - \varepsilon)(1 - 2/\log \log x)$.

For each $\lambda \in \Lambda(U)$, there is an n_λ for which, setting $P_\lambda := \Re(f_\lambda^{n_\lambda}(0))$,

- $P_\lambda < -e^{x+\sqrt{x}} + 2\pi$;
- $|Df_\lambda^{n_\lambda}(0)| < e^{x^9}(C_1/r^K)|P_\lambda|^4$;
- a neighbourhood B_λ of 0 (using Lemma 13) is mapped by some iterate of f_λ onto the $R(U_l, \lambda)$ containing $R(\Lambda^{-1}(\lambda), \lambda)$ and thence onto an element $Q_\lambda := f^{n_\lambda}(B_\lambda)$ of \mathcal{Q} , all with distortion bounded by 2. We have $B(0, 1/|Df_\lambda^{n_\lambda}(0)|) \subset B_\lambda \subset B(0, 1)$.

We note that $e^{P_\lambda/2} > 1/r^{6K} = \exp(6Ke^{x+3})$, so

$$\sup_{Q_\lambda} |f_\lambda| |Df_\lambda^{n_\lambda}(0)|^6 < e^{P_\lambda+2\pi} e^{6x^9} (C_1/r^K)^6 |P_\lambda|^{24} < 1/2,$$

provided x is large. Thus $|Df_\lambda^{n_\lambda+1}|_{|B_\lambda} < |Df_\lambda^{n_\lambda}|_{|B_\lambda}^{-5}$. so

$$f_\lambda^{n_\lambda+1}(B_\lambda) \subset B(f_\lambda^{n_\lambda+1}(0), |Df_\lambda^{n_\lambda}(0)|^{-5}) \subset B(0, |Df_\lambda^{n_\lambda}(0)|^{-4}),$$

so $f_\lambda^{n_\lambda+1}(B_\lambda) \subset B_\lambda$, f_λ has an attracting periodic orbit and λ is a hyperbolic parameter. If H denotes the set of hyperbolic parameters, $H \cap A_j \supset \Lambda(U)$.

Thus $m(H \cap A(\lambda_0; \gamma r, r)) \geq m(\Lambda(U))/m(A_j) \geq (1 - \varepsilon)(1 - 2/\log \log x)$ for all $r < r_1$. Having chosen ε as small as we want, and recalling $r = \exp(-5e^{x+3})$,

$m(H \cap A(\lambda_0; \gamma r, r))$ is arbitrarily close to 1 for all sufficiently small $r > 0$, completing the proof.

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